Folded Bifurcation in Coupled Asymmetric Logistic Maps

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Abstract

A system of coupled two logistic maps, one periodic and the other chaotic, is studied. It is found that with the variation of the coupling strength, the system displays several curious features such as the appearance of quadrupling of period, occurrence of isolated period three attractor and the coexistence of the Hopf and pitchfork bifurcations. Possible applications and extensions are discussed.

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The chaos in higher dimensional system is one of the focal subject of physics today. Along with the approach starting from modeling physical system with many degrees of freedom, there emerged a new approach, developed by Kaneko to couple many one dimensional maps [1-3] to study the behavior of the system as a whole. However, this model can only be applied to study the dynamics of a single medium such as the pattern formation in a fluid. What happens if two media border on each other? One may naturally leads to the model of coupled logistic maps with different strength parameters. Thus it is appropriate to inquire whether loosening the condition of strict identity might bring any new feature while keeping both maps to be logistic to hold the redundant controlling parameters minimal. Even two logistic maps coupled to each other may serve as the dynamical model of driven coupled oscillators. It has been found that two coupled identical maps possess several characteristic features which are typical of higher dimensional chaos [1,4].

In this letter, we report the results of numerical investigation on the system of two logistic maps with different strength parameters such that the one map lies in a period one stable attractor or a bifurcation point and the other in chaotic region when decoupled. Several new features previously unobserved are found. Most notable among them is the appearance of a period four cycle straight from the stable period one cycle. The other peculiar feature is the almost simultaneous occurrence of periods four, eight and sixteen right after the Hopf bifurcation which results in a very intriguing metamorphose of the attractor when one changes the coupling parameter.

The system we study is two linearly coupled maps

$$
\begin{aligned}
x_{n+1} &= (1-\epsilon)f(\mu, x_n) + \epsilon f(\nu, y_n) \\
y_{n+1} &= \epsilon f(\mu, x_n) + (1-\epsilon)f(\nu, y_n)
\end{aligned}
\tag{1}
$$

where the map f is taken to be the logistic map [5] with strength parameters μ.

$$
f(\mu, x) = \mu x(1-x)
\tag{2}
$$

In case of $\mu = \nu$, two maps soon become synchronized no matter what initial conditions may be, *i.e.*, coupled maps are identical with a single logistic map. The interesting is the case of $\mu \neq \nu$. In the following we fix the strength

parameters above and below critical value $\mu^* = 3.56994\ldots$ for μ and ν respectively. We choose the strength parameters μ and ν and regard the coupling parameter ϵ as the controlling parameter.

In Fig. 1, the attractors of the coupled-map are displayed as functions of coupling ϵ. Fig. 1(a) shows the result of $\mu = 4$ and $\nu = 3$, and (b) that of $\mu = 4$ and $\nu = 2$. They are two typical example of various values of μ and ν. One immediately notices several interesting features. The fact that there are two chaotic regions in both $\epsilon = 0$ and $\epsilon = 1$ ends seems odd at first sight, but after some reflection, one realizes that very weak ϵ means very strong $(1 - \epsilon)$ which brings chaos first to the variable x, then to y however weak the coupling term may be. The most salient feature is the appearance of stable period four cycle right after the period one around $\epsilon = 0.77$ in Fig. 1(a). Another, found in both (a) and (b) cases, is the sudden filling of the x and y space around $\epsilon = 0.85$ and above. The broad window-like region with period four around $\epsilon = 0.9$ in the case of (b) is also noteworthy.

Identifying stable and unstable periodic points often gives the skeleton view of the dynamics of the system. With the vector notation of the variables $\vec{z} = (x, y)$ and the operator notation of the map

$$\vec{z}_{n+1} = \vec{F}(\vec{z}_n), \tag{3}$$

one can define the periodic points of cycle N as the solution of the equation

$$\vec{z}_N = \vec{F}^N(\vec{z}_N), \tag{4}$$

where $\vec{F}^N(\vec{z}_N)$ is defined by $\vec{F}^N(\vec{z}_N) = \overbrace{\vec{F}(\vec{F}(\vec{F} \cdots \vec{F}(\vec{z}_N) \cdots))}^{N \ times}$. Bifurcations occur at the parameter value of ϵ where the periodic point satisfies

$$\left| E_{max} \left[\frac{d\vec{F}^N}{d\vec{z}}(\vec{z}_N) \right] \right| = 1, \tag{5}$$

in addition to eq.(4). Here, $E_{max}[G]$ is the maximum eigenvalue of the matrix G. We numerically solve eq.(4) for $N = 3, 4$, and 8 for entire range of $\epsilon = 0 \sim 1$. The results are presented in Fig. 2 where the unstable orbits are shown as well as the stable ones. One can clearly see from $N = 4$ and $N = 8$ skeleton, that the period four occurring at $\epsilon = 0.77$ is the result of the *folded period doubling bifurcation sequence* whose period two starts at $\epsilon = 0.75$ after

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the quadrupling of the period when one moves from period one at larger ϵ to a higher periodic orbits at smaller ϵ. While this type of inversion of the sequence is nothing of unimaginable nature in principle, it has never been observed in simple maps to our knowledge. One recognizes easily that it is rather hard to obtain in a usual one-dimensional map since it requires special tuning of functional dependence of the map on the controlling parameter to ensure that the condition eq.(4) is satisfied for $N = 4$ before $N = 2$. We therefore think it significant that it emerges from nothing but linear coupling of two logistic maps of different strength parameters. One might presume that it should be observed rather frequently in higher dimensional coupled lattice map models once the requirement of strict uniformity of the elements is lifted. In fact, preliminary investigation on three coupled logistic maps supports this view.

There are two notable facts in the periodic cycle diagram above $\epsilon = 0.85$. One is that no periodic point exists in the region $\epsilon < 0.86$. The examination of the $x - y$ profile shows that the onset of "filling area" at approximately $\epsilon = 0.85$ is the result of the Hopf bifurcation[6]. The second is that around $\epsilon = 0.88$, period four and period eight start (as well as other higher periods not shown here) appearing almost simultaneously. While these occurrences fit into the generic transition scenario of "$cycle\ one \rightarrow Hopf \rightarrow cycles \rightarrow \ldots \rightarrow chaos$" found by Kaneko in an early study[1,3], the second fact, crowded onset of many different periodic cycles makes this transition to chaos a very intricate one. These points are visually displayed in the phase portrait of the attractor at several values at and above $\epsilon = 0.85$ shown in Fig. 3. The first two figures (a) and (b) clearly show the Hopf bifurcation around $\epsilon = 0.85$. Subsequent distortion and transformation to an aurora-like strange attractor observed in Fig. 3(c) to (f) certainly match those found in far more complicated system in its aesthetic appeal. One curious feature found both in Figs. 2 and 3(b) is the early appearance of period 3 cycle around $\epsilon = 0.87$. At present, we are unable to notice any direct role of this period to the shape and stability of the attractors around this region[7].

In summary, we have constructed a system consisting of chaotic and periodic maps coupled together. While the dynamics of the system does not go beyond the known bifurcation schemes in the two-dimensional dissipative system (which is certainly not to be expected), it is found that various bifurcations occur in such a combination to give the system several intriguing features.

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Finally, few words on possible applications and extensions are in order. It should not be too difficult to construct a circuit (electronic, for example) to materialize the system proposed here. Actual applicational usage of quadrupling of stable states might be envisioned. Also, replacing the logistic map with other maps, cycle map or quadratic map for example, might be interesting to see the generality and/or the new aspects of the findings here. Another application is possible by increasing the number of maps coupled each other. The coupled lattice map having "impurity", or one element with different strength parameter from all the rest might exhibit non-conventional features. A lattice with alternating strength might also be an interesting system. In a word, loosening the condition of strict uniformity of the elements might bing some new feature to the coupled lattice map.

References

1. K. Kaneko, Prog. Theor. Phys. **69**, 1427 (1983).

2. K. Kaneko, Prog. Theor. Phys. **72**, 480 (1984); *ibid.* **74**, 1033 (1985).

3. K. Kaneko, Collapse of tori and genesis of chaos in dissipative systems (World Scientific, 1986).

4. T. Hogg and B.A. Huberman, Phys. Rev. A**29**, 275 (1984).

5. R. May, Nature **261**, 459 (1976).

6. E. Hopf, Ber. Math.-Phys. K1. Sächs. Akad. Wiss. (Leipzig), **94**, 1 (1942).

7. D. Ruelle and F. Takens, Commun. Math. Phys. **20**, 167 (1971).

Figure Captions

Fig. 1 The phase diagram of the coupled maps eqs.(1)-(2). (a) is for the parameter values $\mu = 4.0$, $\nu = 3.0$, and (b) is for $mu = 4.0$, $\nu = 2.0$.

Fig. 2 The "skelton" of Fig. 1, showing the position of the periodic cycles of the maps. (a) is for $\mu = 4.0$, $\nu = 3.0$, and (b) for $\mu = 4.0$, $\nu = 2.0$ as before. The caption number indicates the period of the cycle.

Fig. 3 Spatial portraits of the attractors for the coupled logistic map with $\mu = 4.0$ and $\nu = 3.0$ at $\epsilon = 0.852$ (a), $\epsilon = 0.868$ (b), $\epsilon = 0.8757$ (c), $\epsilon = 0.877$ (d), $\epsilon = 0.885$ (e), and $\epsilon = 0.900$ (f).

$\mu = 4 \quad \nu = 3$

$\mu = 4 \quad \nu = 2$